Pergamon

Int. J. Solids Structures Vol. 34, No. 20, pp. 2549–2561, 1997

0020-7683/97 $17.00 + .00

PII: S0020-7683(96)00135-7

TEMPERATURE DETERMINATION FOR A CONTACTING BODY BASED ON AN INVERSE PIEZOTHERMOELASTIC PROBLEM

FUMIHIRO ASHIDA
Department of Electronics and Control Engineering, Tsuyama National College of Technology, Tsuyama, Okayama 708, Japan

and

THEODORE R. TAUCHERT
Department of Engineering Mechanics, University of Kentucky, 467 Anderson Hall, Lexington, KY 40506-0046, U.S.A.

(*Received* 25 *January* 1996)

Abstract—The axisymmetric temperature distribution on the surface of a contacting body is inferred from a knowledge of the difference in the electric potential distributions on the two faces of a piezothermoelastic "sensor". The sensor consists of a finite circular disk with hexagonal structure of class 6 mm, having its cylindrical boundary constrained by a rigid, thermally insulated, electrically charge-free ring. The inverse problem is solved using a potential function approach that makes use of three displacement potentials and two electric potentials. Numerical results for a cadmium selenide sensor illustrate the effects of the distribution of the measured electric potential difference and the thickness of the disk upon the heating temperature, elastic displacements, stresses and electric displacements. © 1997 Elsevier Science Ltd.

INTRODUCTION

Piezoelectric materials have been employed in a variety of engineering applications, including ultrasonic transducers and electromechanical actuators. Recently, interest has focused on the use of piezothermoelastic materials as sensors and actuators in smart structural systems undergoing thermomechanical excitations. The usefulness of piezoelectric devices in such systems stems from the coupling that exists between the thermoelastic and electric fields. In sensor applications, thermomechanical disturbances can be determined from measurements of the induced electric potential differences (direct piezoelectric effect); in actuator applications, deformation and/or stress are controlled through the introduction of an appropriate electric field (converse piezo effect).

Whereas most investigations on piezoelasticity have considered isothermal behaviors, a few studies have considered nonisothermal response. Noteworthy are the works of Tiersten (1971), who derived nonlinear equations of thermoelectroelasticity, and Chandrasekharaiah (1988), who developed a generalized linear piezothermoelastic formulation. The present authors proposed a general solution technique for three-dimensional problems of piezothermoelastic hexagonal solids of class 6 mm in cartesian coordinates (Ashida *et al.*, 1994a) and for a two-dimensional problem of an orthotropic plate exhibiting crystal class 2 mm (Ashida *et al.*, 1994b). Also, a solution to asymmetric problems in cylindrical coordinates was developed (Ashida *et al.*, 1994c) and the response of an infinite piezoelectric plate subject to axisymmetric heating was investigated (Ashida *et al.*, 1993). In the present paper we examine the response of a finite circular disk, one face of which is in contact with a heated body. The inverse problem is solved to determine the surface temperature when the electric potential difference between the surfaces of the disk is prescribed. Numerical results illustrate the effects of the distribution of the electric potential difference and the disk thickness on the heating temperature, elastic displacements, stresses and electric displacements.

GOVERNING EQUATIONS

Consider the axisymmetric response of a piezothermoelastic solid of crystal class 6 mm. The constitutive relations for the elastic field, expressed in cylindrical coordinates, are

$$\begin{aligned}
\sigma_{rr} &= c_{11}\varepsilon_{rr}+c_{12}\varepsilon_{\theta\theta}+c_{13}\varepsilon_{zz}-e_1E_z-\beta_1T \\
\sigma_{\theta\theta} &= c_{12}\varepsilon_{rr}+c_{11}\varepsilon_{\theta\theta}+c_{13}\varepsilon_{zz}-e_1E_z-\beta_1T \\
\sigma_{zz} &= c_{13}\varepsilon_{rr}+c_{13}\varepsilon_{\theta\theta}+c_{33}\varepsilon_{zz}-e_3E_z-\beta_3T \\
\sigma_{zr} &= c_{44}\varepsilon_{zr}-e_4E_r
\end{aligned} \tag{1}$$

in which σ_{ij} are the stress components, ε_{ij} are the strain components, E_i are the electric field intensities, T denotes the temperature rise, c_{ij} are elastic stiffnesses, e_i are piezoelectric constants and β_i are stress-temperature coefficients. The strains are related to the displacements u_i as follows:

$$\varepsilon_{rr} = u_{r,r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}u_r, \quad \varepsilon_{zz} = u_{z,z}, \quad \varepsilon_{zr} = u_{z,r}+u_{r,z}. \tag{2}$$

The constitutive equations for the electric field are

$$D_r = e_4\varepsilon_{zr}+\eta_1E_r, \quad D_z = e_1\varepsilon_{rr}+e_1\varepsilon_{\theta\theta}+e_3\varepsilon_{zz}+\eta_3E_z+p_3T \tag{3}$$

where D_i are the electric displacement components, η_i are dielectric permitivities and p_3 is the pyroelectric constant.

The equations of equilibrium are

$$\sigma_{rr,r}+\sigma_{rz,z}+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta}) = 0, \quad \sigma_{zr,r}+\sigma_{zz,z}+\frac{1}{r}\sigma_{zr} = 0 \tag{4}$$

and the equation of electrostatics is

$$D_{r,r}+D_{z,z}+\frac{1}{r}D_r = 0 \tag{5}$$

while the equation governing stationary heat conduction is

$$\Delta_1T+\lambda^2T_{,zz} = 0 \tag{6}$$

where

$$\Delta_1 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}, \quad \lambda^2 = \frac{\lambda_z}{\lambda_r} \tag{7}$$

in which λ_i are coefficients of heat conduction. For the problem considered here it is convenient to represent the temperature as the sum of two functions, namely

$$T = T_0(z)+T_1(r,z). \tag{8}$$

SOLUTION PROCEDURE

In a previous investigation (Ashida *et al.*, 1993), the authors presented a method of solution applicable in the case of the temperature field $T_1(r,z)$. The formulation entails a piezothermoelastic potential function (ϕ), three piezoelastic potential functions (ψ_i, i = 1–

3) and an electric potential (Φ_1). We introduce here an additional piezothermoelastic potential ($\Psi_0(z)$) and an electric potential ($\Phi_0(z)$) associated with the one-dimensional temperature field $T_0(z)$. The displacements and electric field intensities are related to these potential functions through the relations

$$u_r = (\Psi_1 + \Psi_2)_{,r}, \quad u_z = (\Psi_0 + k_1\Psi_1 + j\Psi_2)_{,z} \tag{9}$$

and

$$E_r = -\Phi_{,r}, \quad E_z = -\Phi_{,z} \tag{10}$$

where (Ashida *et al.*, 1993)

$$\Psi_1 = \phi + \psi_1 + \psi_2 + \psi_3 \tag{11}$$

$$\Psi_{0,zz} = \gamma_1 T_0 \tag{12}$$

$$\Psi_{2,zz} = \frac{1}{\xi_1}(\Delta_1\Psi_1 + v_1\Psi_{1,zz} - \delta_1 T_1) \tag{13}$$

and

$$\Phi = \Phi_0 + \Phi_1 \tag{14}$$

$$\Phi_{0,z} = \gamma_2 T_0 \tag{15}$$

$$\Phi_{1,z} = \frac{1}{\xi_2}(\Delta_1\Psi_2 + v_2\Psi_{2,zz} - \delta_2 T_1). \tag{16}$$

The equations of equilibrium (4) and electrostatics (5) are then satisfied, providing the potential functions ϕ and ψ_i satisfy the equations (see Ashida *et al.*, 1993)

$$\left(\Delta_1 + \mu_1\frac{\partial^2}{\partial z^2}\right)\left(\Delta_1 + \mu_2\frac{\partial^2}{\partial z^2}\right)\left(\Delta_1 + \mu_3\frac{\partial^2}{\partial z^2}\right)\phi = d_2\Delta_1\Delta_1 T_1 + d_1\Delta_1 T_{1,zz} + d_0 T_{1,zzzz} \tag{17}$$

and

$$\Delta_1\psi_i + \mu_i\psi_{i,zz} = 0 \quad (i = 1\text{–}3). \tag{18}$$

The coefficients appearing in eqns (9)–(18) are given by

$$k_i = \frac{c_{11}v_i - c_{44}}{c_{13}+c_{44}}, \quad j = k_1 - \frac{k_1 v_2 c_{11}}{c_{44}} + \frac{c_{11}(e_3 - \eta e_4)}{(e_1+e_4)(c_{44}+e_4^2/\eta_1)}, \quad \eta = \frac{\eta_3}{\eta_1}$$

$$\gamma_1 = \frac{\beta_3\eta_3 - e_3 p_3}{c_{33}\eta_3 + e_3^2}, \quad \gamma_2 = \frac{c_{33}p_3 + \beta_3 e_3}{c_{33}\eta_3 + e_3^2}$$

$$\xi_1 = (c_{13}+c_{44})\left[\frac{v_1 k_2}{c_{44}} - \frac{e_3 - \eta e_4}{(e_1+e_4)(c_{44}+e_4^2/\eta_1)}\right], \quad \xi_2 = -\frac{e_1+e_4}{c_{11}}$$

$$\delta_1 = \frac{\beta_1}{c_{11}} - \delta_2, \quad \delta_2 = \frac{(e_1+e_4)[k_1 v_2(c_{44}+e_4^2/\eta_1)\beta_1 - c_{44}(\beta_3 - p_3 e_4/\eta_1)]}{c_{11}[(c_{44}+e_4^2/\eta_1)(e_1+e_4)k_1 v_2 - c_{44}(e_3 - \eta e_4)]}$$

$$d_0 = \eta(v_2\delta_1 + \xi_1\delta_2) - b_2\xi_2\delta_1 + \frac{p_3\xi_1\xi_2}{\eta_1}$$

$$d_1 = v_2\delta_1 + \xi_1\delta_2 + \eta\delta_1 - a_2\xi_2\delta_1, \quad d_2 = \delta_1 \tag{19}$$

where v_1 and v_2 are roots of the equation

$$c_{11}\left(c_{44}+\frac{e_4^2}{\eta_1}\right)v^2+\left[c_{13}^2+2c_{13}c_{44}-c_{11}c_{33}+\frac{(c_{13}+c_{44})(e_1+e_4)e_4-c_{11}e_3e_4-c_{44}e_4^2}{\eta_1}\right]v$$
$$+c_{44}\left(c_{33}+\frac{e_3e_4}{\eta_1}\right)=0 \quad (20)$$

and μ_1, μ_2 and μ_3 are roots of

$$\mu^3-(v_1+v_2+\eta-a_2\xi_2)\mu^2+[v_1v_2+\eta(v_1+v_2)-(v_1a_2+b_2)\xi_1-a_1\xi_1\xi_2]\mu$$
$$-\eta v_1v_2+v_1b_2\xi_2+b_1\xi_1\xi_2=0 \quad (21)$$

in which

$$a_1=\frac{e_1+(1+k_1)e_4}{\eta_1},\quad b_1=\frac{k_1e_3}{\eta_1},\quad a_2=\frac{e_1+(1+j)e_4}{\eta_1},\quad b_2=\frac{je_3}{\eta_1}. \quad (22)$$

INVERSE PROBLEM FOR PIEZOTHERMOELASTIC SENSOR

Consider a circular piezothermoelastic disk (Fig.1) having one face in contact with a heated body. We wish to determine the temperature distribution $T(r,b)=\Theta(r)$ on the surface of the contacting body, from a knowledge of the difference in the electric potential across the faces of the disk. The cylindrical boundary of the disk is constrained by a rigid, thermally insulated and electrically charge-free ring. For this situation the boundary conditions are taken to be

$$T=0 \quad \text{on } z=-b \quad (23)$$

$$\sigma_{rz}=\sigma_{zz}=D_z=0 \quad \text{on } z=\pm b \quad (24)$$

$$T_{,r}=u_r=D_r=0 \quad \text{on } r=a. \quad (25)$$

It is presumed that a potential difference of the form

$$[\Phi]_{z=b}-[\Phi]_{z=-b}=V_0v(r) \quad (26)$$

is measured, where V_0 is a constant having dimension of electric potential and $v(r)$ describes the radial variation of the potential difference.

A general solution to the heat conduction eqn (6) is

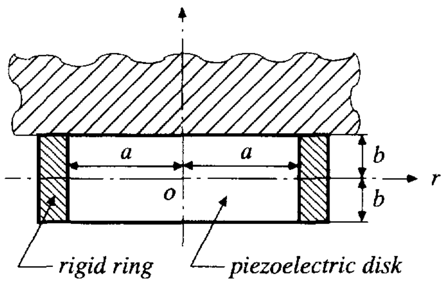

Fig. 1. Piezoelectric sensor.

$$T = A_0 + B_0 z + \sum_{n=1}^{\infty} J_0(\alpha_n r)\left[A_n \cosh\frac{\alpha_n z}{\lambda} + B_n \sinh\frac{\alpha_n z}{\lambda}\right] \tag{27}$$

in which A_n and B_n ($n = 0$–∞) are unknown coefficients. Note, for consistency with eqn (8), $T_0 = A_0 + B_0 z$, while T_1 constitutes the infinite series in eqn (27).

Edge condition $T_{,r} = 0$ on $r = a$ is satisfied providing $\alpha_n a$ are roots of

$$J_1(\alpha_n a) = 0. \tag{28}$$

By substituting T_1 into eqn (17), the piezothermoelastic potential function ϕ is found to be

$$\phi = \sum_{n=1}^{\infty} F_{1n} J_0(\alpha_n r)\left[A_n \cosh\frac{\alpha_n z}{\lambda} + B_n \sinh\frac{\alpha_n z}{\lambda}\right] \tag{29}$$

where

$$F_{1n} = \frac{\lambda^2(d_2\lambda^4 - d_1\lambda^2 + d_0)}{\alpha_n^2(\mu_1 - \lambda^2)(\mu_2 - \lambda^2)(\mu_3 - \lambda^2)}. \tag{30}$$

The piezoelastic potentials ψ_i, which represent solutions to eqn (18), are

$$\psi_i = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left[K_{in} \cosh\frac{\alpha_n z}{\sqrt{\mu_i}} + L_{in} \sinh\frac{\alpha_n z}{\sqrt{\mu_i}}\right] \quad (i = 1\text{–}3) \tag{31}$$

where K_{in} and L_{in} are unknown coefficients. The displacement function Ψ_1, obtained by substituting eqns (29) and (31) into (11), then becomes

$$\Psi_1 = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left\{F_{1n}\left[A_n \cosh\frac{\alpha_n z}{\lambda} + B_n \sinh\frac{\alpha_n z}{\lambda}\right] + \sum_{i=1}^{3}\left[K_{in} \cosh\frac{\alpha_n z}{\sqrt{\mu_i}} + L_{in} \sinh\frac{\alpha_n z}{\sqrt{\mu_i}}\right]\right\}. \tag{32}$$

Likewise, substitution of T_0 into eqn (12) leads to

$$\Psi_0 = \gamma_1\left(A_0\frac{z^2}{2} + B_0\frac{z^3}{6}\right) \tag{33}$$

whereas substitution of T_1 and Ψ_1 into eqn (13) yields

$$\Psi_2 = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left\{F_{2n}\left[A_n \cosh\frac{\alpha_n z}{\lambda} + B_n \sinh\frac{\alpha_n z}{\lambda}\right] + \sum_{i=1}^{3}(\ell_i - 1)\left[K_{in} \cosh\frac{\alpha_n z}{\sqrt{\mu_i}} + L_{in} \sinh\frac{\alpha_n z}{\sqrt{\mu_i}}\right]\right\} \tag{34}$$

where

$$F_{2n} = \frac{(v_1 - \lambda^2)F_{1n} - \lambda^2 \delta_1/\alpha_n^2}{\xi_1}, \quad \ell_i = 1 + \frac{v_1 - \mu_i}{\xi_1} = 1 + \frac{m_i - k_1}{j}. \tag{35}$$

The electric potentials Φ_0 and Φ_1, found by substituting T_0 and T_1 into eqns (15) and (16), become, after integration

$$\Phi_0 = \gamma_2\left(A_0 z + B_0 \frac{z^2}{2}\right) \tag{36}$$

and

$$\Phi_1 = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left\{F_{3n}\left[A_n \sinh\frac{\alpha_n z}{\lambda} + B_n \cosh\frac{\alpha_n z}{\lambda}\right] + \sum_{i=1}^{3} \frac{n_i \alpha_n}{\sqrt{\mu_i}}\left[K_{in} \sinh\frac{\alpha_n z}{\sqrt{\mu_i}} + L_{in} \cosh\frac{\alpha_n z}{\sqrt{\mu_i}}\right]\right\} \tag{37}$$

in which

$$F_{3n} = \frac{\alpha_n(v_2 - \lambda^2)F_{2n} - \lambda^2 \delta_2/\alpha_n}{\lambda \xi_2}, \quad n_i = \frac{(v_1 - \mu_i)(v_2 - \mu_i)}{\xi_1 \xi_2}. \tag{38}$$

Substitution of eqns (32)–(34), (36) and (37) into (9), (10), (2), (1) and (3) yields, respectively, the following expressions for the displacements, electric field intensities, stresses and electric displacements:

$$u_r = -\sum_{n=1}^{\infty} \alpha_n J_1(\alpha_n r)\left\{(F_{1n} + F_{2n})\left[A_n \cosh\frac{\alpha_n z}{\lambda} + B_n \sinh\frac{\alpha_n z}{\lambda}\right] + \sum_{i=1}^{3} \ell_i\left[K_{in} \cosh\frac{\alpha_n z}{\sqrt{\mu_i}} + L_{in} \sinh\frac{\alpha_n z}{\sqrt{\mu_i}}\right]\right\} \tag{39}$$

$$u_z = \gamma_1\left(A_0 z + B_0 \frac{z^2}{2}\right) + \sum_{n=1}^{\infty} \alpha_n J_0(\alpha_n r)\left\{\left(\frac{k_1 F_{1n} + j F_{2n}}{\lambda}\right)\left[A_n \sinh\frac{\alpha_n z}{\lambda} + B_n \cosh\frac{\alpha_n z}{\lambda}\right] + \sum_{i=1}^{3} \frac{m_i}{\sqrt{\mu_i}}\left[K_{in} \sinh\frac{\alpha_n z}{\sqrt{\mu_i}} + L_{in} \cosh\frac{\alpha_n z}{\sqrt{\mu_i}}\right]\right\} \tag{40}$$

$$E_r = \sum_{n=1}^{\infty} \alpha_n J_1(\alpha_n r)\left\{F_{3n}\left[A_n \sinh\frac{\alpha_n z}{\lambda} + B_n \cosh\frac{\alpha_n z}{\lambda}\right] + \sum_{i=1}^{3} \frac{n_i \alpha_n}{\sqrt{\mu_i}}\left[K_{in} \sinh\frac{\alpha_n z}{\sqrt{\mu_i}} + L_{in} \cosh\frac{\alpha_n z}{\sqrt{\mu_i}}\right]\right\} \tag{41}$$

$$E_z = -\gamma_2(A_0 + B_0 z) - \sum_{n=1}^{\infty} \alpha_n J_0(\alpha_n r)\left\{\frac{F_{3n}}{\lambda}\left[A_n \cosh\frac{\alpha_n z}{\lambda} + B_n \sinh\frac{\alpha_n z}{\lambda}\right] + \sum_{i=1}^{3} \frac{n_i \alpha_n}{\mu_i}\left[K_{in} \cosh\frac{\alpha_n z}{\sqrt{\mu_i}} + L_{in} \sinh\frac{\alpha_n z}{\sqrt{\mu_i}}\right]\right\} \tag{42}$$

$$\sigma_{rr} = (c_{13}\gamma_1 + e_1\gamma_2 - \beta_1)(A_0 + B_0 z) + \sum_{n=1}^{\infty}\left(\left\{\left[-c_{11}(F_{1n} + F_{2n})\alpha_n^2 + c_{13}(k_1 F_{1n}\right.\right.\right.$$

$$+jF_{2n})\frac{\alpha_n^2}{\lambda^2}+e_1F_{3n}\frac{\alpha_n}{\lambda}-\beta_1\Bigg]J_0(\alpha_n r)+(c_{11}-c_{12})(F_{1n}+F_{2n})\frac{\alpha_n J_1(\alpha_n r)}{r}\Bigg\}\left[A_n\cosh\frac{\alpha_n z}{\lambda}\right.$$

$$\left.+B_n\sinh\frac{\alpha_n z}{\lambda}\right]-\sum_{i=1}^{3}\alpha_n^2\left[\left(c_{11}\ell_i-\frac{c_{13}m_i+e_1n_i}{\mu_i}\right)J_0(\alpha_n r)-(c_{11}-c_{12})\ell_i\frac{J_1(\alpha_n r)}{\alpha_n r}\right]$$

$$\times\left[K_{in}\cosh\frac{\alpha_n z}{\sqrt{\mu_i}}+L_{in}\sinh\frac{\alpha_n z}{\sqrt{\mu_i}}\right]\Bigg). \tag{43}$$

$$\sigma_{\theta\theta}=(c_{13}\gamma_1+e_1\gamma_2-\beta_1)(A_0+B_0z)+\sum_{n=1}^{\infty}\Bigg(\Bigg\{\Bigg[-c_{12}(F_{1n}+F_{2n})\alpha_n^2+c_{13}(k_1F_{1n}$$

$$+jF_{2n})\frac{\alpha_n^2}{\lambda^2}+e_1F_{3n}\frac{\alpha_n}{\lambda}-\beta_1\Bigg]J_0(\alpha_n r)-(c_{11}-c_{12})(F_{1n}+F_{2n})\frac{\alpha_n J_1(\alpha_n r)}{r}\Bigg\}\left[A_n\cosh\frac{\alpha_n z}{\lambda}\right.$$

$$\left.+B_n\sinh\frac{\alpha_n z}{\lambda}\right.-\sum_{i=1}^{3}\alpha_n^2\left[\left(c_{12}\ell_i-\frac{c_{13}m_i+e_1n_i}{\mu_i}\right)J_0(\alpha_n r)+(c_{11}-c_{12})\ell_i\frac{J_1(\alpha_n r)}{\alpha_n r}\right]$$

$$\times\left[K_{in}\cosh\frac{\alpha_n z}{\sqrt{\mu_i}}+L_{in}\sinh\frac{\alpha_n z}{\sqrt{\mu_i}}\right]\Bigg). \tag{44}$$

$$\sigma_{zz}=\sum_{n=1}^{\infty}J_0(\alpha_n r)\Bigg\{\left[-c_{13}(F_{1n}+F_{2n})\alpha_n^2+c_{33}(k_1F_{1n}+jF_{2n})\frac{\alpha_n^2}{\lambda^2}+e_3F_{3n}\frac{\alpha_n}{\lambda}-\beta_3\right]$$

$$\times\left[A_n\cosh\frac{\alpha_n z}{\lambda}+B_n\sinh\frac{\alpha_n z}{\lambda}\right]-\sum_{i=1}^{3}\alpha_n^2\left[\left(c_{13}\ell_i-\frac{c_{33}m_i+e_3n_i}{\mu_i}\right)\right.$$

$$\left.\times\left[K_{in}\cosh\frac{\alpha_n z}{\sqrt{\mu_i}}+L_{in}\sinh\frac{\alpha_n z}{\sqrt{\mu_i}}\right]\right\}. \tag{45}$$

$$\sigma_{rz}=-\sum_{n=1}^{\infty}J_1(\alpha_n r)\Bigg(\left\{c_{44}\left[(1+k_1)F_{1n}+(1+j)F_{2n}\right]\frac{\alpha_n^2}{\lambda}+e_4F_{3n}\alpha_n\right\}$$

$$\times\left[A_n\sinh\frac{\alpha_n z}{\lambda}+B_n\cosh\frac{\alpha_n z}{\lambda}\right]+\sum_{i=1}^{3}\alpha_n^2\frac{c_{44}(\ell_i+m_i)+e_4n_i}{\sqrt{\mu_i}}\times\left[K_{in}\sinh\frac{\alpha_n z}{\sqrt{\mu_i}}+L_{in}\cosh\frac{\alpha_n z}{\sqrt{\mu_i}}\right]\Bigg). \tag{46}$$

$$D_r=-\sum_{n=1}^{\infty}J_1(\alpha_n r)\Bigg(\left\{e_4\left[(1+k_1)F_{1n}+(1+j)F_{2n}\right]\frac{\alpha_n^2}{\lambda}-\eta_1F_{3n}\alpha_n\right\}$$

$$\times\left[A_n\sinh\frac{\alpha_n z}{\lambda}+B_n\cosh\frac{\alpha_n z}{\lambda}\right]+\sum_{i=1}^{3}\alpha_n^2\frac{e_4(\ell_i+m_i)-\eta_1n_i}{\sqrt{\mu_i}}\times\left[K_{in}\sinh\frac{\alpha_n z}{\sqrt{\mu_i}}+L_{in}\cosh\frac{\alpha_n z}{\sqrt{\mu_i}}\right]\Bigg). \tag{47}$$

$$D_z=\sum_{n=1}^{\infty}J_0(\alpha_n r)\Bigg\{\left[-e_1(F_{1n}-F_{2n})\alpha_n^2+e_3(k_1F_{1n}+jF_{2n})\frac{\alpha_n^2}{\lambda^2}-\eta_3F_{3n}\frac{\alpha_n}{\lambda}+p_3\right]$$

$$\times\left[A_n\cosh\frac{\alpha_n z}{\lambda}+B_n\sinh\frac{\alpha_n z}{\lambda}\right]-\sum_{i=1}^{3}\alpha_n^2\left[e_1\ell_i-\frac{e_3m_i-\eta_3n_i}{\mu_i}\right]$$

$$\times\left[K_{in}\cosh\frac{\alpha_n z}{\sqrt{\mu_i}}+L_{in}\sinh\frac{\alpha_n z}{\sqrt{\mu_i}}\right]\Bigg\}. \tag{48}$$

The $8n+2$ unknown coefficients $(A_0, B_0, A_n, B_n, K_{1n}, K_{2n}, K_{3n}, L_{1n}, L_{2n}$ and $L_{3n})$ $(n = 1-\infty)$ in these equations are determined through application of the boundary conditions. Conditions (25) are identically satisfied by expressions (27), (39) and (47). Condition (23) requires that

$$A_0 - B_0 b = 0, \quad A_n \cosh\frac{\alpha_n b}{\lambda} - B_n \sinh\frac{\alpha_n b}{\lambda} = 0. \tag{49}$$

The boundary conditions (24) governing the stresses σ_{rz} and σ_{zz} and electric displacement D_z, imply, respectively,

$$\left\{c_{44}\left[(1+k_1)F_{1n}+(1+j)F_{2n}\right]\frac{\alpha_n^2}{\lambda}+e_4F_{3n}\alpha_n\right\}\left[A_n\sinh\frac{\alpha_n b}{\lambda}, B_n\cosh\frac{\alpha_n b}{\lambda}\right]$$
$$+\sum_{i=1}^{3}\alpha_n^2\frac{c_{44}(\ell_i+m_i)+e_4n_i}{\sqrt{\mu_i}}\left[K_{in}\sinh\frac{\alpha_n b}{\sqrt{\mu_i}}, L_{in}\cosh\frac{\alpha_n b}{\sqrt{\mu_i}}\right] = 0, \tag{50}$$

$$\left[-c_{13}(F_{1n}+F_{2n})\alpha_n^2+c_{33}(k_1F_{1n}+jF_{2n})\frac{\alpha_n^2}{\lambda^2}+e_3F_{3n}\frac{\alpha_n}{\lambda}-\beta_3\right]$$
$$\times\left[A_n\cosh\frac{\alpha_n b}{\lambda}, B_n\sinh\frac{\alpha_n b}{\lambda}\right]-\sum_{i=1}^{3}\alpha_n^2\left[c_{13}\ell_i-\frac{c_{33}m_i+e_3n_i}{\mu_i}\right]$$
$$\times\left[K_{in}\cosh\frac{\alpha_n b}{\sqrt{\mu_i}}, L_{in}\sinh\frac{\alpha_n b}{\sqrt{\mu_i}}\right] = 0 \tag{51}$$

and

$$\left[-e_1(F_{1n}+F_{2n})\alpha_n^2+e_3(k_1F_{1n}+jF_{2n})\frac{\alpha_n^2}{\lambda^2}-\eta_3F_{3n}\frac{\alpha_n}{\lambda}+p_3\right]$$
$$\times\left[A_n\cosh\frac{\alpha_n b}{\lambda}, B_n\sinh\frac{\alpha_n b}{\lambda}\right]-\sum_{i=1}^{3}\alpha_n^2\left[e_1\ell_i-\frac{e_3m_i-\eta_3n_i}{\mu_i}\right]$$
$$\times\left[K_{in}\cosh\frac{\alpha_n b}{\sqrt{\mu_i}}, L_{in}\sinh\frac{\alpha_n b}{\sqrt{\mu_i}}\right] = 0. \tag{52}$$

Furthermore, expressing the radial variation $v(r)$ of the electric potential (see eqn (26)) as

$$v(r) = \bar{v}_0 + \sum_{n=1}^{\infty}\bar{v}_nJ_0(\alpha_n r) \tag{53}$$

where

$$\bar{v}_0 = \frac{2}{a^2}\int_0^a rv(r)\,\mathrm{d}r, \quad \bar{v}_n = \frac{2}{a^2J_0^2(\alpha_n a)}\int_0^a rv(r)J_0(\alpha_n r)\,\mathrm{d}r \tag{54}$$

then condition (26) requires that

$$\left[\gamma_2A_0b, \quad F_{3n}A_n\sinh\frac{\alpha_n b}{\lambda}+\sum_{i=1}^{3}\frac{n_i\alpha_n}{\sqrt{\mu_i}}K_{in}\sinh\frac{\alpha_n b}{\sqrt{\mu_i}}\right] = \frac{V_0}{2}[\bar{v}_0, \bar{v}_n]. \tag{55}$$

Equations (49)–(52) and (55) are sufficient to determine all unknown coefficients in the solution. Substitution of coefficients A_n and B_n ($n = 0$–∞) into eqn (27) then yields the desired heating temperature distribution

$$\Theta(r) = A_0 + B_0 b + \sum_{n=1}^{\infty} J_0(\alpha_n r)\left[A_n \cosh\frac{\alpha_n b}{\lambda} + B_n \sinh\frac{\alpha_n b}{\lambda}\right]. \tag{56}$$

NUMERICAL EXAMPLE

As an illustrative example, it is assumed that the measured electric potential difference $[\Phi]_{z=b} - [\Phi]_{z=-b}$ across the piezoelectric sensor has a distribution described by

$$v(r) = -\left(1 - 2f\frac{r^2}{a^2} + f\frac{r^4}{a^4}\right) \tag{57}$$

in which f is a specified parameter. The sensor material is considered to be cadmium selenide, having the following properties (Berlincourt *et al.*, 1963):

$$\begin{aligned}
c_{11} &= 74.1\times10^9\,\mathrm{Nm^{-2}}, c_{12} = 45.2\times10^9\,\mathrm{Nm^{-2}}, c_{13} = 39.3\times10^9\,\mathrm{Nm^{-2}},\\
c_{33} &= 83.6\times10^9\,\mathrm{Nm^{-2}}, c_{44} = 13.2\times10^9\,\mathrm{Nm^{-2}},\\
\beta_1 &= 0.621\times10^6\,\mathrm{NK^{-1}\,m^{-2}}, \beta_3 = 0.551\times10^6\,\mathrm{NK^{-1}\,m^{-2}},\\
e_1 &= -0.160\,\mathrm{Cm^{-2}}, e_3 = 0.347\,\mathrm{Cm^{-2}}, e_4 = -0.138\,\mathrm{Cm^{-2}},\\
\eta_1 &= 82.6\times10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}}, \eta_3 = 90.3\times10^{-12}\,\mathrm{C^2\,N^{-1}\,m^{-2}},\\
p_3 &= -2.94\times10^{-6}\,\mathrm{CK^{-1}\,m^{-2}}, Y_r = 42.8\times10^9\,\mathrm{Nm^{-2}},\\
\alpha_r &= 4.4\times10^{-6}\,\mathrm{K^{-1}}, d_1 = -3.92\times10^{-12}\,\mathrm{CN^{-1}}
\end{aligned} \tag{58}$$

where Y_r is Young's modulus, α_r is the coefficient of linear thermal expansion and d_1 is the piezoelectric coefficient. Since the values of the coefficients of heat conduction for cadmium selenide could not be found in the literature, the value $\lambda^2 = \lambda_z/\lambda_r = 1.5$ is assumed. For this material, the roots of eqns (20) and (21) are

$$\begin{aligned}
&v_1 = 0.3405, \quad v_2 = 3.2334\\
&\mu_1 = 0.2987, \quad \mu_2 = 1.4490, \quad \mu_3 = 2.8452.
\end{aligned} \tag{59}$$

For convenience in presentation of numerical results, the following dimensionless quantities are introduced:

$$\begin{aligned}
&\bar{b} = \frac{b}{a}, \quad \bar{r} = \frac{r}{a}, \quad \bar{z} = \frac{z}{a}, \quad \bar{\Theta} = \frac{a\alpha_r\Theta}{|d_1|V_0}, \quad \bar{u}_i = \frac{u_i}{|d_1|V_0},\\
&\bar{\sigma}_{ij} = \frac{a\sigma_{ij}}{|d_1|Y_r V_0}, \quad \bar{\Phi} = \frac{\Phi}{V_0}, \quad \bar{E}_i = \frac{aE_i}{V_0}, \quad \bar{D}_i = \frac{aD_i}{d_1^2 Y_r V_0}.
\end{aligned} \tag{60}$$

Response quantities were calculated by retaining the first 50 terms in each of the

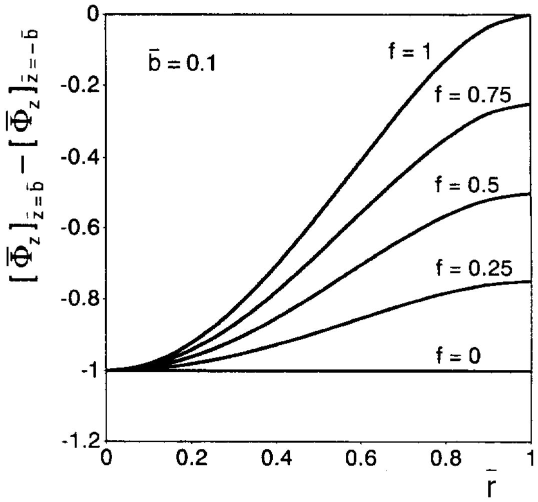

Fig. 2. Distributions of electric potential for different values of parameter f; $\bar{b} = 0.1$.

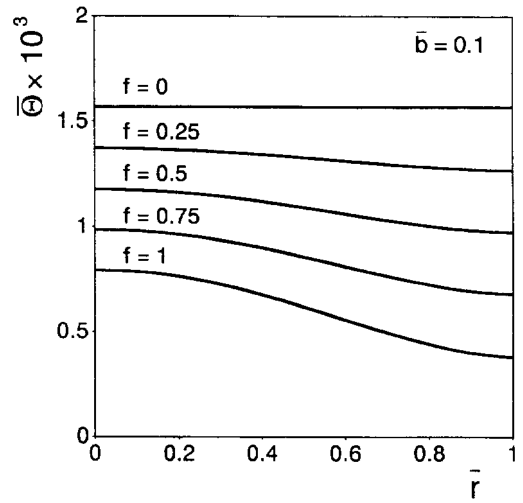

Fig. 3. Distributions of inferred surface temperature for different values of parameter f; $\bar{b} = 0.1$.

corresponding infinite series. Figure 2 shows the nondimensionalized electric potential difference across a disk of thickness $\bar{b} = 0.1$, for various values of the parameter f. The corresponding inferred temperature distributions $\bar{\Theta}(\bar{r})$ are shown in Fig. 3. Note that a uniform measured potential difference $[\bar{\Phi}]_{\bar{z}=\bar{b}} - [\bar{\Phi}]_{\bar{z}=-\bar{b}} = -1$ (case of $f = 0$) implies a constant surface temperature $\bar{\Theta}$. In dimensional terms, it follows that a uniform potential difference $V_0 = 1$ volt across a disk of thickness $2b = 2$ mm, for example, implies a temperature rise $\Theta = 0.141$ K.

Figures 4–6 show the associated distributions of elastic displacements, stresses and electric displacements for this example. For the constant temperature case ($f = 0$) it is found that $\bar{u}_r = \bar{D}_r = 0$, while the quantities $\bar{u}_z$, $\bar{\sigma}_{rr}$ and $\bar{\sigma}_{\theta\theta}$ are independent of $\bar{r}$.

The influence of sensor thickness $\bar{b}$ on the inferred surface temperature, displacements, stresses and radial electric displacement is shown in Figs 7–10. The radial elastic displacement, as well as the radial and circumferential stresses on the bottom surface are not plotted, since their values are negligibly small compared with the corresponding upper surface values. For a given amplitude V_0 of electric potential, the thinner the disk (smaller $\bar{b}$), the larger the associated (absolute) values of $\bar{\Theta}$, $\bar{u}_r$, $\bar{\sigma}_{rr}$ and $\bar{\sigma}_{\theta\theta}$ on the upper surface.

The results obtained here indicate that piezoelectric materials can indeed be utilized to measure surface temperature distributions, providing further evidence that piezoelectric elements can be used effectively in intelligent structural systems.

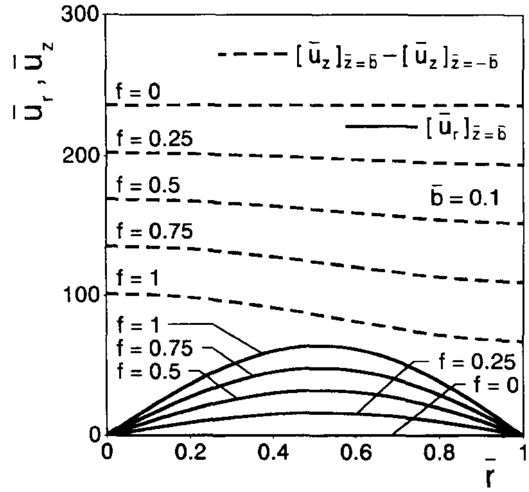

Fig. 4. Distributions of radial and axial displacements for different values of parameter f; $\bar{b} = 0.1$.

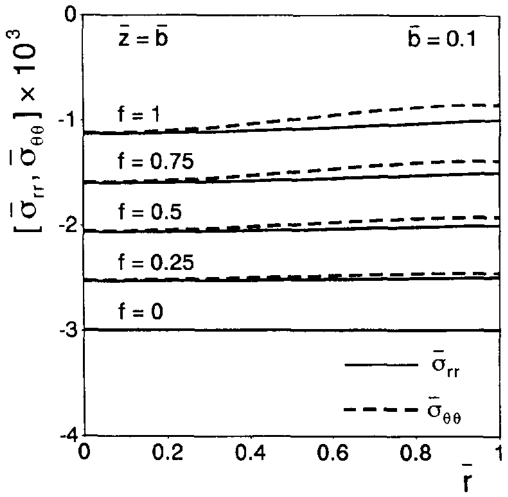

Fig. 5. Distributions of radial and circumferential stresses for different values of parameter f; $\bar{b} = 0.1$.

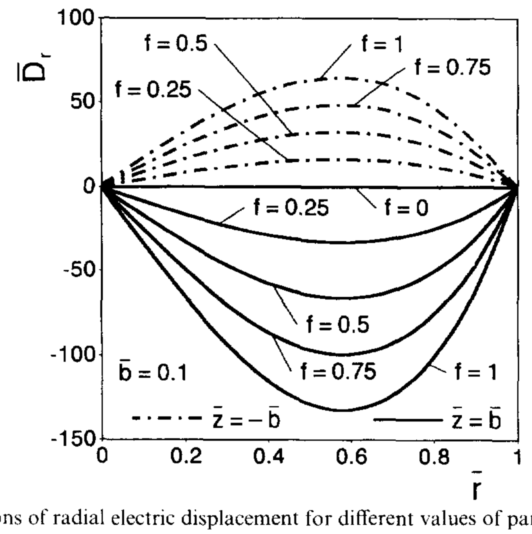

Fig. 6. Distributions of radial electric displacement for different values of parameter f; $\bar{b} = 0.1$.

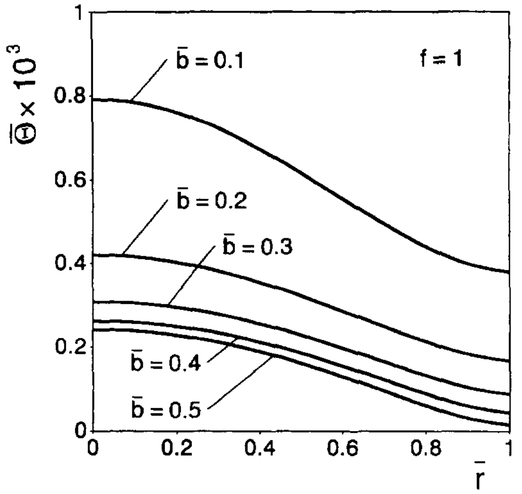

Fig. 7. Effect of sensor thickness $\bar{b}$ on inferred temperature; $f = 1$.

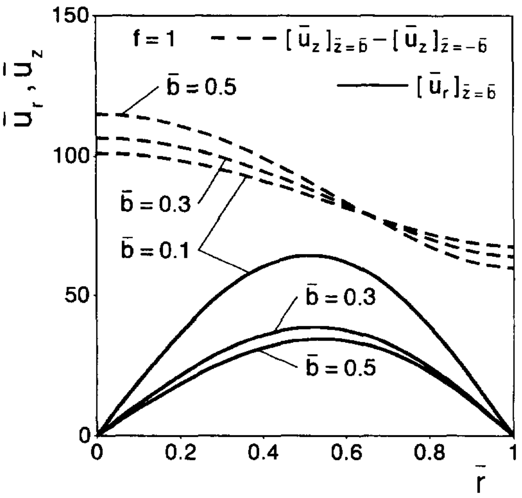

Fig. 8. Effect of sensor thickness $\bar{b}$ on radial and axial displacements; $f = 1$.

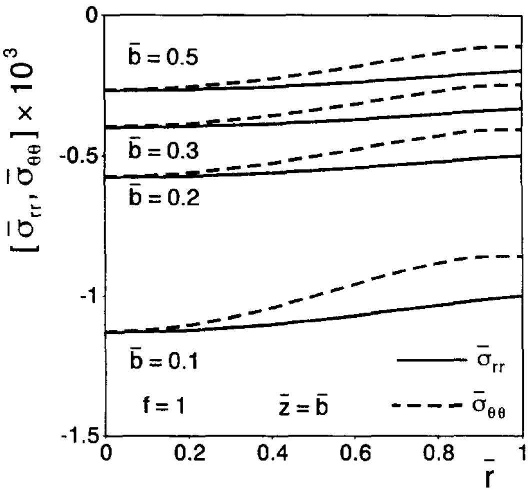

Fig. 9. Effect of sensor thickness $\bar{b}$ on radial and circumferential stresses; $f = 1$.

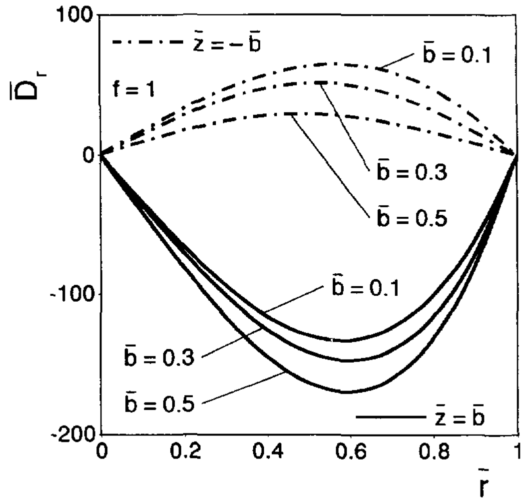

Fig. 10. Effect of sensor thickness $\bar{b}$ on radial electric displacement ; $f = 1$.

REFERENCES

Ashida, F., Tauchert, T. R. and Noda, N. (1994a). A general solution technique for piezothermoelasticity of hexagonal solids of class 6 mm in cartesian coordinates, *Zeitschrift Angewandt Math. Mech.* **74,** 87–95.

Ashida, F., Noda, N. and Tauchert, T. R. (1994b). A two-dimensional piezothermoelastic problem in an orthotropic plate exhibiting crystal class 2 mm. *JSME International Journal* **37,** 334–340.

Ashida, F., Tauchert, T. R. and Noda, N. (1994c). Potential function method for piezothermoelastic problems of solids of crystal class 6 mm in cylindrical coordinates. *Journal of Thermal Stresses* **17,** 361–375.

Ashida, F., Tauchert, T. R. and Noda, N. (1993). Response of a piezothermoelastic plate of crystal class 6 mm subject to axisymmetric heating. *International Journal of Engineering Science* **31,** 373–384.

Berlincourt, D., Jaffe, H. and Shiozawa, L. R. (1963). Electroelastic properties of the sulfides, selenides, and tellurides of zinc and cadmium. *Physics Review* **129,** 1009–1017.

Chandrasekharaiah, D. S. (1988). A generalized linear thermoelasticity theory for piezoelectric media. *Acta Mechanica* **71,** 39–49.

Tiersten, H. F. (1971). On the nonlinear equations of thermoelectroelasticity. *International Journal of Engineering Science* **9,** 587–604.